

Quasi-Parallel Segments and Characterization of Unique Bichromatic Matchings*

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Abstract

Given n red and n blue points in general position in the plane, it is well-known that there is a perfect matching formed by non-crossing line segments. We characterize the bichromatic point sets which admit exactly one non-crossing matching. We give several geometric descriptions of such sets, and find an $O(n \log n)$ algorithm that checks whether a given bichromatic set has this property.

1 Introduction

Basic notation and preliminary results A *bichromatic* $(n + n)$ point set F is a set of n blue points and n red points in the plane. We assume that the points of F are in general position, this is, no three points lie on the same line. A *perfect bichromatic non-crossing straight-line matching* of F is a set of n non-crossing segments between points of F so that each blue point is connected to exactly one red point, and vice versa. Following the convention in the literature, we call such matchings *BR-matchings*. For notational reasons, we shall denote the colors blue and red by \circ (white) and \bullet (black).

It is well known and easy to see that any F has at least one BR-matching. One way to see this is to use recursively the Ham-Sandwich Theorem (see Theorem 32 below); another way is to show that the bichromatic matching that minimizes the total length of segments is necessarily non-crossing. The main motivation of our work is to characterize bichromatic sets with a *unique* BR-matching. We will establish connections between this question and various other geometric notions that have shown up in different contexts.

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In what follows, M denotes a given BR-matching of F . The segments in M are considered directed from the \circ -end to the \bullet -end. For $A \in M$, the line that contains A is denoted by $g(A)$, and it is considered directed consistently with A . For two directed segments A and B for which the lines $g(A)$ and $g(B)$ do not cross, we say that the segments (resp., the lines) are *parallel* if they have the same orientation; otherwise we call them *antiparallel*. If we delete inner points of A from $g(A)$, we obtain two closed *outer rays*: the \circ -ray and the \bullet -ray, according to the endpoint of A that belongs to the ray.

The convex hull of F will be denoted by $\text{CH}(F)$, and its boundary by $\partial\text{CH}(F)$. Consider the circular sequence of colors of the points of $\partial\text{CH}(F)$; a *color interval* is a maximal subsequence of this circular sequence that consists of points of the same color. In the point set in Fig. 1(a), $\partial\text{CH}(F)$ has four color intervals: two \circ -intervals (of sizes 1 and 2) and two \bullet -intervals (of sizes 2 and 3).

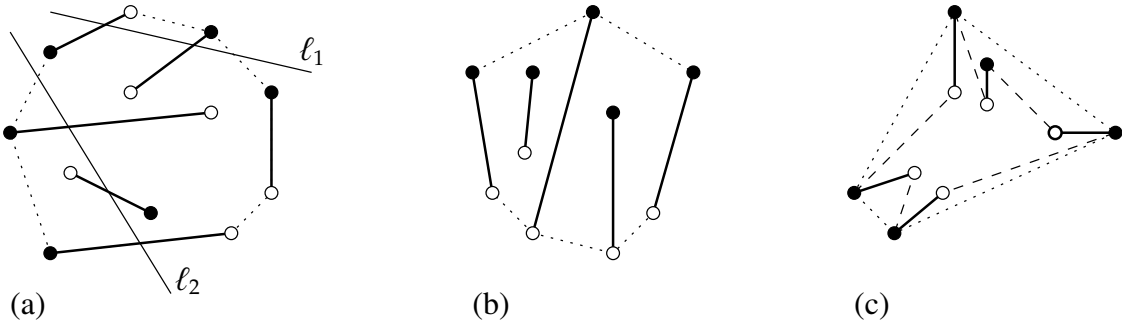


Figure 1: (a) A matching with chromatic cuts; (b) A linear matching; (c) A circular matching (another matching for the same point set is indicated by dashed lines).

In order to state our main result, we need the notion of chromatic cut. A *chromatic cut* of M is a line ℓ that crosses two segments of M such that their \bullet -ends are on different sides of ℓ (ℓ can as well cross other segments of M). For example, the lines ℓ_1 and ℓ_2 in Fig. 1(a) are chromatic cuts. The matchings in Fig. 1(b) and (c) have no chromatic cuts. Aloupis et al. [3, Lemma 9] proved that a BR-matching M that has a chromatic cut cannot be unique. They actually proved a stronger statement: there is even a *compatible* BR-matching $M' \neq M$, which means that the union of M and M' is non-crossing. Thus, having no chromatic cut is a *necessary condition* for a unique BR-matching. (We will give a simpler proof of this fact in Section 2, without establishing the existence of a compatible matching.) However, it is *not sufficient*, as shown by the example in Fig. 1(c).

We will give a thorough treatment of BR-matchings without chromatic cuts. We shall prove in Lemma 7 that BR-matchings without chromatic cuts can be classified into the following two types. A *matching of linear type* (or, for shortness, *linear matching*) is a BR-matching without a chromatic cut such that $\partial\text{CH}(F)$ consists of exactly two color intervals (both necessarily of size at least 2). A *matching of circular type* (or *circular matching*) is a BR-matching without a chromatic cut such that all points of $\partial\text{CH}(F)$ have the same color. The reason for these terms will be clarified below. Fig. 1(b–c) shows a linear and circular matching.

We shall prove that the unique BR-matchings are precisely the linear matchings.

This will be a part of our main result, Theorem 2 below.

The main result. We formulate the main result in the following three Theorems.

Theorem 1. *Let M be a BR-matching without a chromatic cut. Then M is either of linear or circular type.*

We will characterize both linear and circular matchings.

The following series of equivalent characterizations of unique BR-matchings. Condition 3 refers to a relation \triangleleft which is defined later in Definition 8. The notion of a *quasi-parallel matching* in Condition 5 will be defined in Definition 12. We state the conditions here to have all equivalent characterizations in one place.

Theorem 2 (Characterization of unique BR-matchings). *Let M be a BR-matching of F . 2 Then the following conditions are equivalent:*

1. M is the only BR-matching of F .
2. M is a linear matching.
3. The relation \triangleleft is a linear order on M .
4. No subset of segments forms one of the three forbidden patterns shown below in Fig. 2.
5. M is quasi-parallel.

Moreover, if M satisfies any of the above conditions, then any submatching of M satisfies the above conditions. This follows from the fact that conditions 3–5 directly imply that they hold for all subsets. (This is most trivial for condition 4.)

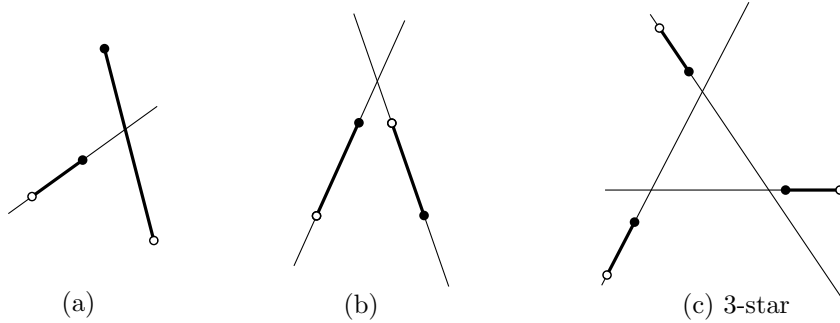


Figure 2: Forbidden patterns for quasi-parallel matchings.

Theorem 3 (Properties of circular matchings). *Let M be a BR-matching of F . Then the following conditions are equivalent:*

1. M is a circular matching.
2. The sidedness relation \triangleleft is total but not a linear order.

3. No subset of segments forms one of the forbidden patterns shown in Fig. 2 (a)-(b) and at least three segments form a 3-star as in Fig. 2 (c).

Furthermore, if these conditions hold, then:

- p1. The sidedness relation \triangleleft induces naturally a circular order, explained further in Section 5.
p2. There are (at least) two additional disjoint BR-matchings M' and M'' on F .

Property p1 justifies the term circular matching.

	Linear Type	Circular Type
Uniqueness	M is unique	M is not unique
Patterns from Fig. 2	(a), (b) and (c) are avoided	(a) and (b) are avoided; (c) is present
Relation \triangleleft	Linear order	Total, not linear; induces a circular order

Related work. Monochromatic and bichromatic straight-line matchings have been intensively studied in the recent years.

One direction is *geometric augmentation*. Given a matching, one wants to determine whether it is possible to add segments in order to get a bigger matching with a certain property, under what conditions can this be done, how many segments one has to add, etc. See Hurtado and Cs. D. Tóth [6] for a recent survey.

The *bichromatic compatible matching graph* of F has at its node set the BR-matchings of F . Two BR-matchings are joined by an edge if they are compatible. Aloupis, Barba, Langerman and Souvaine [3] proved recently that the bichromatic compatible matching graph is always connected.

For non-colored point sets, one can speak about the (*monochromatic*) *compatible matching graph*. The diameter of this graph is $O(\log n)$ [1], whereas for the bichromatic compatibility graph, no non-trivial upper bound is known. García, Noy, and Tejel [5] showed that the number of perfect monochromatic matchings is minimized among all n -point sets when the points are in convex position. Ishaque, Souvaine, and Cs. Tóth [7] showed that for any monochromatic perfect matching, there is a *disjoint* monochromatic compatible matching.

Other related work involves counting the maximal number of BR-matchings that a point set admits. Sharir and Welzl [13] established a bound of $O(7.61^n)$.

Outline. In Section 2 we prove in a simple way that if a BR-matching M has a chromatic cut, then it is not unique. In Section 4 we give different characterizations of linear matchings, and we prove that a linear matching is unique. In Section 5 we analyze circular matchings, and prove that they are never unique. In Section 6 we complete the proof of Theorem 2. In Section 7 we discuss parallelizability of linear matchings and enumerate sidedness relations realizable by n element circular matchings. In Section 8 we describe an algorithm that recognizes point sets F which admit exactly one matching, an algorithm that recognizes circular matchings, and some more algorithms that compute some other notions we used in the previous Sections. We conclude by mentioning some open problems and possible directions for future research in Section 9.

2 Chromatic cuts

We start with a simple geometric description of BR-matchings that admit a chromatic cut.

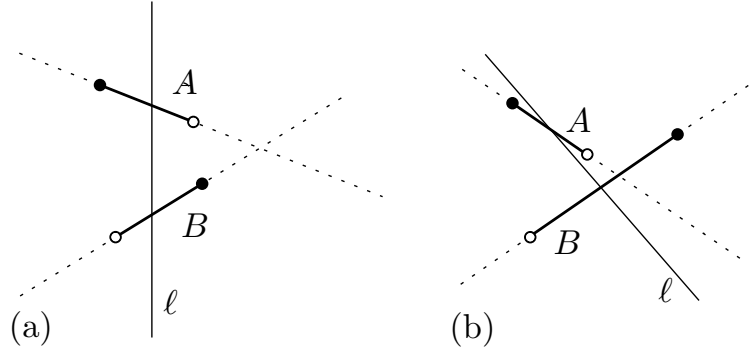


Figure 3: (a) Outer rays of different colors intersect; (b) An outer ray crosses the second segment.

Lemma 4. *Let M be a BR-matching of F . M admits a chromatic cut if and only if there are two segments $A, B \in M$ such that A and B are antiparallel, or the intersection point of $g(A)$ and $g(B)$ belongs to outer rays of different colors, or an outer ray of one of the segments crosses the second segment.*

Proof. “ \Leftarrow ” If the ●-ray of one segment and the ○-ray of the second segment cross each other, then any line through inner points of A and B is a chromatic cut, see Fig. 3 (a). The same is true if A and B are antiparallel. If an outer ray of one of the segments crosses the second segment, then, if we rotate $g(A)$ around an inner point of A by a small angle in one of two possible directions (depending on the orientation of A and B), then a chromatic cut is obtained, see Fig. 3 (b).

“ \Rightarrow ” Let ℓ be a chromatic cut of M , and let A and B be two segments that have their ●-ends on the opposite sides of ℓ . Consider the lines $g(A)$ and $g(B)$. If $g(A)$ and $g(B)$ don’t cross, they clearly must be antiparallel. If they cross, then it is not possible that the two outer rays of the same color meet, because they are on opposite sides of ℓ . ☺

A line ℓ is a *balanced line* if in each open halfplane determined by ℓ , the number of ●-points is equal to the number of ○-points. We say that an open halfplane is *dominated* by ●-points if it contains more ●-points than ○-points. The next lemma reveals a relation between chromatic cuts and balanced lines.

Lemma 5. *Let M be a BR-matching. M has a chromatic cut if and only if there exists a balanced line that crosses a segment of M .*

Proof. “ \Leftarrow ” Let ℓ be a balanced line that crosses a segment A . We can assume that ℓ does not contain points from F : it cannot contain exactly one point of F ; and if it contains two points of F of different colors, we can translate it slightly, obtaining a balanced line that still crosses A but does not contain points of F . If it contains two points of the same color, we rotate it slightly about the midpoint between these two points.

Now, A has a \bullet -end in one half-plane of ℓ and a \circ -end in the other half-plane. Since ℓ is balanced, there must be at least one other segment B that has its \bullet -end and \circ -end on the opposite sides as A . So, ℓ is a chromatic cut.

" \Rightarrow " First, let A be a segment in M , and let p be an inner point of A that does not belong to any line determined by two points of F , other than the endpoints of A . We claim that if there is no balanced line that crosses A at p , then $g(A)$ is the only balanced line through p .

Assume that there is no balanced line that crosses A at p . We use a continuity argument. Let $m = m_0$ be any directed line that crosses A in p . Rotate m around p counterclockwise until it makes a half-turn. Denote by m_α the line obtained from m after rotation by the angle α ; so, we rotate it until we get m_π . Let φ ($0 < \varphi < \pi$) be the angle such that m_φ coincides with $g(A)$. Assume without loss of generality that the right halfplane bounded by m is dominated by \bullet ; then the right halfplane bounded by m_π is dominated by \circ . As we rotate m , the points of F change the side *one by one*, except at $\alpha = \varphi$. When one point changes sides, m_α cannot change from \bullet -dominance to \circ -dominance without becoming a balanced line. Therefore, for each $0 \leq \alpha < \varphi$, the right side of m_α is dominated by \bullet , and for each $\varphi < \alpha \leq \pi$, the right side of m_α is dominated by \circ . At $\alpha = \varphi$, exactly two points of different colors change sides. The only possibility is that the \bullet -end of A passes from the right side to the left side and the \circ -end of A passes from the left side to the right side of the rotated line. It follows that at this moment the value of $\#(\bullet) - \#(\circ)$ in the right halfplane changes from 1 to -1 , and that $m_\varphi = g(A)$ is a balanced line.

Now, let ℓ be a chromatic cut that crosses $A, B \in M$ so that the \bullet -end of A and the \circ -end of B are in the same half-plane bounded by ℓ . Denote by p and q the points of intersection of ℓ with A and B , respectively. We assume without loss of generality that p and q do not belong to any line determined by points of F .

If there is a balanced line that crosses A at p , or a balanced line that crosses B at q , we are done. By the above claim, it remains to consider the case when the only balanced line through p is $g(A)$ and the only balanced line through q is $g(B)$. Assume without loss of generality that ℓ is horizontal, p is left of q , and the \bullet -end of A is above ℓ , see Fig. 4 for an illustration.

We start with the line $k = g(A)$, directed upwards, rotate it clockwise around p until it coincides with ℓ , and then continue to rotate it clockwise around q until it coincides with $g(B)$, directed down. We monitor $\#(\bullet) - \#(\circ)$ on the right side of the line k as above: this quantity is 0 in the initial and the final position. Just after the initial position it is -1 , and just before the final position it is $+1$. In between, it makes only ± 1 jumps, since the points of F change the side of the rotated line k one by one. It follows that for some intermediate position it is 0 — a contradiction. \odot

In Section 8, we discuss the algorithmic implementation of this proof.

Corollary 6. *Let M be a BR-matching of F with a chromatic cut. Then there is a different matching $M' \neq M$.*

Proof. By Lemma 5, there exists a balanced line ℓ crossing a segment $A \in M$. We construct matchings on both sides of ℓ , and denote their union by M' . Then M' is a matching of F , and we have $M' \neq M$ since M' does not use A . \odot

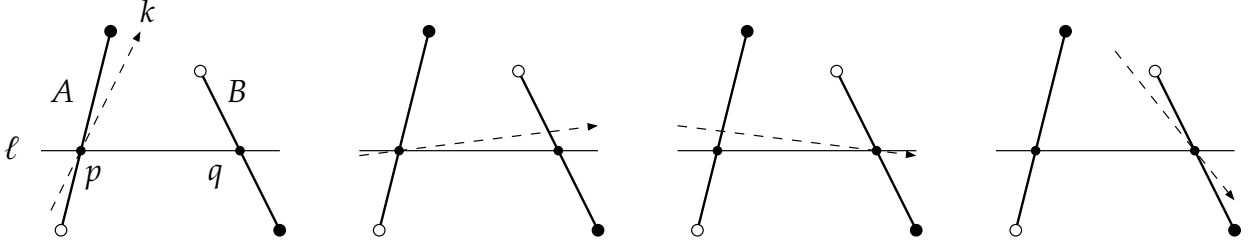


Figure 4: Illustration to the proof of Lemma 5.

Remark. As mentioned in the introduction, Corollary 6 follows from the stronger statement of [3, Lemma 9]: the existence of a compatible matching $M' \neq M$. We have given a simpler alternative proof.

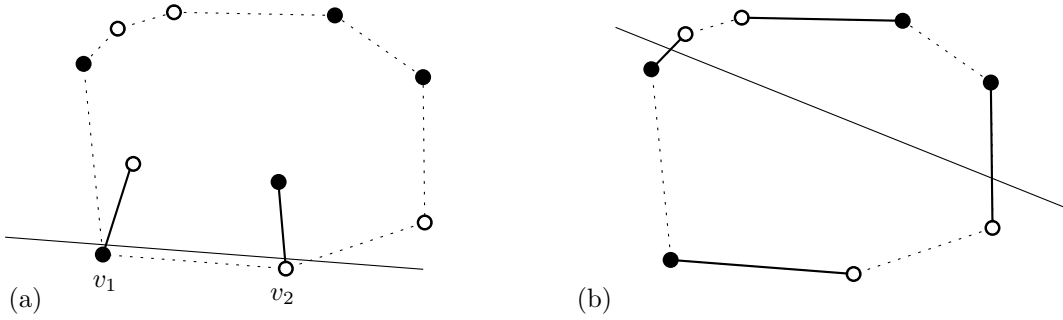


Figure 5: Illustration to the proof of Lemma 7.

Lemma 7. *Let M be a BR-matching of F that has no chromatic cut. Then*

- *either all points of $\partial CH(F)$ have the same color,*
- *or the points of $\partial CH(F)$ form two color intervals of size at least 2.*

In the latter case, the two boundary segments connecting points of different color necessarily belong to M .

Proof. Assume that $\partial CH(F)$ has points of both colors.

If v_1 and v_2 are two neighboring points on $\partial CH(F)$ with different colors, then they are matched by a segment of M . Indeed, let ℓ' be the line through v_1 and v_2 . If v_1 and v_2 are not matched by a segment of M , then each of them is an endpoint of some segment of M . When we shift ℓ' slightly so that it crosses these two segments, a chromatic cut is obtained, see Fig. 5 (a).

Therefore, if the points of $\partial CH(F)$ form more than two color intervals, then at least four segments of M have both ends on $\partial CH(F)$. At least two among them have the \bullet -end before the \circ -end, with respect to their circular order. Any line that crosses these two segments will be then a chromatic cut, see Fig. 5 (b).

Thus, we have exactly two color intervals. If one of them consists of one point, then this point has two neighbors of another color of $\partial CH(F)$. As observed above, this point must be matched by M to both of them, which is clearly impossible. \odot

We recall the notation from the introduction: a *linear matching* is a BR-matching without a chromatic cut such that $\partial\text{CH}(F)$ consists of exactly two color intervals, both of size at least 2; a *circular matching* is a BR-matching without a chromatic cut such that all points of $\partial\text{CH}(F)$ have the same color. So, we proved now that if a BR-matching has no chromatic cut, then it necessarily belongs to one of these types. In the next sections we study these types in more detail.

3 A Sidedness Relation between Segments

Definition 8. For two segments A, B , we define the *sidedness relation* \triangleleft as follows: $A \triangleleft B$ if B lies strictly right of $g(A)$ and A lies strictly left of $g(B)$.

Lemma 9. Let M be a BR-matching. M has no chromatic cut iff the sidedness relation \triangleleft is a total relation, i. e., for any two segments $A, B \in M$, $A \neq B$, we have $A \triangleleft B$ or $B \triangleleft A$.

Proof. If two segments have a chromatic cut, then their supporting lines must intersect as in Fig. 2 (a) or (b), and the segments are not comparable by \triangleleft ; otherwise, the supporting lines are parallel or intersect in the outer rays of the same color, and then the segments are comparable. \odot

Note that the relation \triangleleft is asymmetric by definition: we never have $A \triangleleft B$ and $B \triangleleft A$. Moreover, if M has no chromatic cut, then, in order to prove $A \triangleleft B$, it suffices to prove only one condition from the definition.

Lemma 10. Let M be a BR-matching without chromatic cut; $A, B \in M$ ($A \neq B$). If B lies right of $g(A)$ or A lies left of $g(B)$ then $A \triangleleft B$.

Proof. If M has no chromatic cut by Lemma 9 either $A \triangleleft B$ or $B \triangleleft A$. If we know one of the above conditions, $B \triangleleft A$ is ruled out. \odot

4 Quasi-Parallel, or Linear, Matchings

In this section we give several characterizations of linear matchings and prove that such matchings are unique for their point sets.

Lemma 11. If M is a linear matching, then it has a minimum and a maximum element with respect to \triangleleft .

Proof. By Lemma 7, the two boundary segments connecting points of different color belong to M . For one of them (to be denoted by A_1), all other segments of M belong to the right half-plane bounded by $g(A_1)$; for the second (to be denoted by A_n), all other segments of M belong to the left half-plane bounded by $g(A_n)$. Since M has no chromatic cut, it follows from Lemma 10 that A_1 is the minimum, and A_n is the maximum element of M with respect to \triangleleft . \odot

Definition 12. A BR-matching M is called *quasi-parallel* if there exists a directed reference line ℓ such that the following conditions hold:

- (i) No segment is perpendicular to ℓ .
- (ii) For any $A \in M$, the direction of its projection on ℓ (as usual, from \circ to \bullet) coincides with the direction of ℓ .
- (iii) For any non-parallel $A, B \in M$, the projection of the intersection point of $g(A)$ and $g(B)$ on ℓ lies outside the convex hull of the projections of A and B on ℓ .

The notion of quasi-parallel segments was introduced by Rote [12, 11] as a generalization of parallel segments, in the context of a dynamic programming algorithm for some instances of the traveling salesman problem. In that work, the segments were uncolored; thus, our definition is a “colored” version of the original one. Fig. 6 shows an example of quasi-parallel matching, with horizontal ℓ .

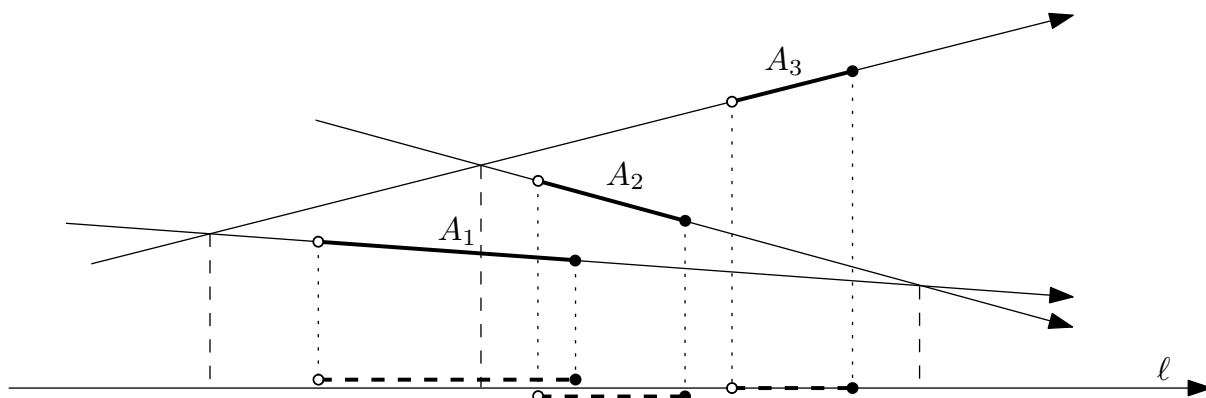


Figure 6: A quasi-parallel matching.

Fig. 2. shows three “patterns” — configurations of two or three segments with respect to the intersection pattern of their supporting lines (thus, they can be expressed in terms of order types). The patterns are considered up to exchanging the colors and reflection. A pair of antiparallel segments is considered as a special case of the pattern (b).

Lemma 13. *Let M be a BR-matching of a bichromatic $(n + n)$ set F . Then the following conditions are equivalent:*

1. M is a linear matching.
2. The relation \triangleleft in M is a strict linear order.
3. M has no patterns of the three kinds in Fig. 2.
4. M is a quasi-parallel matching.

Proof. “1 \Rightarrow 2” By definition, the relation \triangleleft is asymmetric, and according to Lemma 9, it is total.

It remains to prove transitivity. As a linear matching, M has a minimum A_1 and a maximum A_n with respect to \triangleleft by Lemma 11. We define inductively A_2, \dots, A_{n-1} as

follows. Assume A_1, \dots, A_{i-1} are already defined. Remove A_1, \dots, A_{i-1} from M . Then the new matching is again of linear type. It has still no chromatic cut and still both colors on the boundary of the convex hull because A_n belongs to it. Denote the new minimum element by A_i and repeat.

Note that A_j lies to the right of $g(A_i)$, $\forall i < j$, by construction. Thus $A_i \triangleleft A_j$, $\forall i < j$, by Lemma 10. This implies transitivity of \triangleleft .

“2 \Rightarrow 3” It is easy to check that none of the configurations in Fig. 2 is ordered linearly by \triangleleft .

“3 \Rightarrow 4” In this proof, we follow the idea from [11, 12]. As a preparation, one can establish by case distinction that any two or three segments which contain none of the patterns from Fig. 2 are quasi-parallel.

Now, let M be a BR-matching without the forbidden patterns in Fig. 2. For each $A \in M$, let $a(A)$ be the arc on the circle of directions corresponding to positive directions of lines m such that the angle between A and m is acute. (These are the lines that can play the role of a reference line ℓ in the definition of quasi-parallel matching, with respect to A .) Each $a(A)$ is an open half-circle, see Fig. 7.

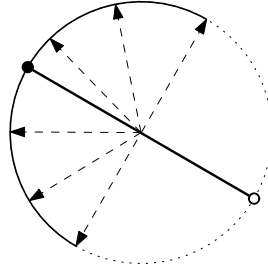


Figure 7: The open arc $a(A)$ for a matching segment A , used in the proof of Lemma 13, 3 \Rightarrow 4.

Fix a segment S of M . For any segments $A \in M$. $\{S, A\}$ is a quasi-parallel matching, and hence the intersection of the corresponding arcs $a(S) \cap a(A)$ is a non-empty subinterval of $a(S)$, which we denote by $a'(A)$. Now, for any two segments $A, B \in M$, $\{S, A, B\}$ is a quasi-parallel matching, and hence the intersection of the corresponding arcs is non-empty. In other words, $a'(A) \cap a'(B) \neq \emptyset$. We can apply Helly's Theorem to the intervals $a'(A)$ (considering them as one-dimensional subintervals of $a(S)$) and conclude that there exists a direction in the intersection of the arcs corresponding to all segments of M . A line ℓ in this direction will fulfill conditions (i) and (ii) of the definition of quasi-parallel matching. Finally, the absence of forbidden patterns implies that condition (iii) is satisfied as well.

“4 \Rightarrow 1” Condition (iii) in the definition of quasi-parallel matchings implies that for any $A, B \in M$, $A \neq B$, the lines $g(A)$ and $g(B)$ are either parallel, or the outer rays of the same color cross. It follows from Lemma 4 that there is no chromatic cut.

The highest point and the lowest point of F , with respect to ℓ , belong to the boundary of the convex hull and have different colors. Therefore, M is of linear type. \odot

Remark. The characterization of quasi-parallel matchings by forbidden patterns was found earlier by Rote in [11]. We have fewer forbidden patterns, because we deal with

colored segments. (In the journal version [12] of this result, the list of patterns was incomplete: the pattern corresponding to Fig. 2 (c) had been overlooked.)

Lemma 13 proves the equivalence of conditions 2, 3, 4, and 5 in Theorem 2. Condition 3 justifies the term “matching of linear type”. Now we prove that they imply the uniqueness of M .

Theorem 14. *Let M be a linear matching on the point set F . Then M is the only matching of F .*

Proof. By Lemma 13, the matching M is quasi-parallel, with reference line ℓ . We assume without loss of generality that ℓ is vertical.

Assume for contradiction that another matching M' exists. (In the figures below, the segments of M are denoted by solid lines, and the segments of M' by dashed lines.) The symmetric difference of M and M' is the union of alternating cycles. We now claim that *an alternating cycle must intersect itself*.

Consider the alternating cycle $\Pi = p_1q_1p_2q_2p_3q_3 \dots p_1$ that consists of segments of $p_iq_i \in M$ and $q_ip_{i+1} \in M'$. We assume that p_i are \circ -vertices and q_i are \bullet -vertices. Let B be the minimum (with respect to \triangleleft) segment and let C be the maximum segment of M that belongs to Π . Then no points of Π lie left of $g(B)$ or right of $g(C)$. Since for both B and C the \bullet -end is higher than the \circ -end, the path Π must cross itself at least once, establishing the claim, see Fig. 8.

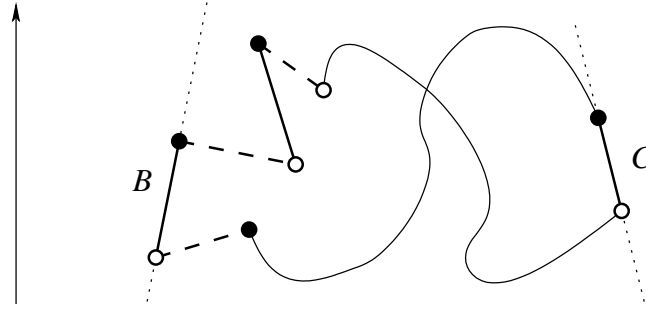


Figure 8: Illustration for the proof of Theorem 14: an alternating path for M crosses itself.

We now traverse the path Π , starting from $p_1q_1p_2q_2 \dots$, until it crosses itself for the first time, say, in a point r . There can be no crossing r between two segments of M or two segments of M' . Hence, the first occurrence of r on Π is on a segment p_iq_i of M , and the second is on a segment q_ip_{i+1} of M' , or vice versa. We consider only the first case, the other being similar. In this case, we consider the matching N that consists of segments $rq_i, p_{i+1}q_{i+1}, p_{i+2}q_{i+2}, \dots, p_jq_j$ (that is, N consists of the segments of M that occur on Π between the two times that it visits r , and the part of segment of M that contains r). It is clear that N is also quasi-parallel.

The closed path $rq_ip_{i+1}q_{i+1}p_{i+2}q_{i+2} \dots p_jq_jr$ is an alternating path for N . By the choice of r , this path does not intersect itself, see Fig. 9, which contradicts the claim proved above that an alternating path of a quasi-parallel matching always intersects itself. \odot

The proof we have just given established the uniqueness of M directly. A weaker version of Theorem 14 was known before [11, 12, Lemma 2]: for a linear matching M ,

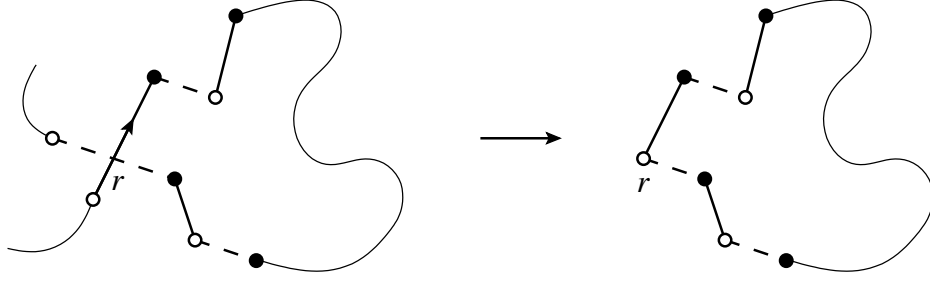


Figure 9: Illustration to the proof of Theorem 14: an alternating path for N .

there is no other *compatible* matching $M' \neq M$. This implies Theorem 14 by the fact that the compatible matching graph is connected [3].

The proof of Theorem 14 tells us that a closed alternating path cannot exist. In contrast, it is always possible to construct two *open* alternating paths from the minimum to the maximum element of M :

Lemma 15. *Let M be a linear matching. Then there exist two alternating paths containing all segments of M , appearing according to the order \triangleleft .*

Proof. Let A_1, \dots, A_n be the segments of M , ordered by \triangleleft . We proceed by induction. Let R_k be a path from A_1 to A_k in which the segments of M appear according to \triangleleft . We obtain R_{k+1} by taking R_k and adding a color-conforming segment from A_k to A_{k+1} . This is possible because there is no other segment of M between A_k and A_{k+1} . The color of the starting point can be chosen and thus we have two such paths, see Fig. 10. ☺

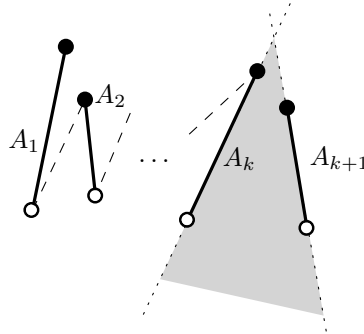


Figure 10: Illustration to the proof of Lemma 15.

Proof of Theorem 14 by the Fishnet Lemma. We will give another proof, which captures the intuition that one gets when drawing an alternating path and trying to close it. Indeed, when one starts to construct an alternating path as in the first proof, one quickly gets the feeling of being stuck: even though it is permitted that segments of M and M' may cross, one cannot close the path because one is forced in one direction. This feeling can be made precise with the following *Fishnet Lemma*. We will apply it only to polygonal curves, but we formulate it for arbitrary curves, see Fig. 11.

Consider a set $V = \{v_1, \dots, v_m\}$ of pairwise noncrossing unbounded Jordan curves (“ropes”). They partition the plane into $m + 1$ connected regions. We assume that they are numbered in such a way that in going from v_i to v_j ($j > i + 1$), one has to cross $v_{i+1}, v_{i+2}, \dots, v_{j-1}$. These curves will be called the *vertical* curves. In the illustrations, they will be black, and we think of them as numbered from left to right.

Consider another set $G = \{g_1, \dots, g_n\}$ of pairwise noncrossing Jordan arcs, called the *horizontal* arcs and drawn in green, such that every curve g_k has its endpoints on two different vertical curves v_i and v_j ($j > i$), has exactly one intersection point with each vertical curve $v_i, v_{i+1}, v_{i+2}, \dots, v_j$, and no intersection with the other curves. See Fig. 11 (a) for an example. We say that the curves $V \cup G$ form a *partial (combinatorial) grid*.

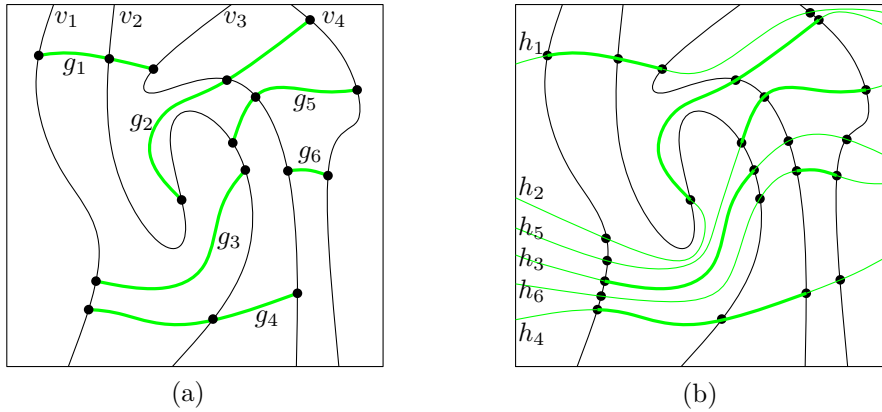


Figure 11: (a) A partial grid. (b) Extension to a full grid of ropes.

Lemma 16 (The Fishnet Lemma). *The horizontal arcs g_k of a partial combinatorial grid $V \cup G$ can be extended to pairwise noncrossing unbounded Jordan arcs h_k in such a way that the curves $H = \{h_1, \dots, h_n\}$ together with V form a full combinatorial grid $V \cup H$: each horizontal curve h_k crosses each vertical curve v_i exactly once. See Fig. 11 (b).*

Proof. This is an easy construction, which incrementally grows the horizontal segments.

The bounded faces of the given curve arrangement $V \cup G$ are quadrilaterals: they are bounded by two consecutive vertical curves and two horizontal curves.

The bounded faces of the desired final curve arrangement $V \cup H$ are also such quadrilaterals, with the additional property that they have no extra vertices on their boundary besides the four corner intersections. In $V \cup G$, such extra vertices arise as the endpoints of the segments g_k .

Let us take such a bounded face, between two vertical curves v_i and v_{i+1} , with an endpoint of g_k on one of its vertical sides, see Fig. 12 (a)–(b). We can easily extend g_k to some point on the opposite vertical side, chosen to be distinct from all other endpoints, splitting the face into two and creating a new intersection point. (The existence of such an extension follows from the Jordan–Schoenflies Theorem, by which the bounded face is homeomorphic to a disc.) An unbounded face between two successive vertical curves v_i and v_{i+1} that has an extra vertex on a vertical side can be treated similarly.

We continue the above extension procedure as long as possible. Since we are adding new intersection points, but no two curves can intersect twice, this must terminate.

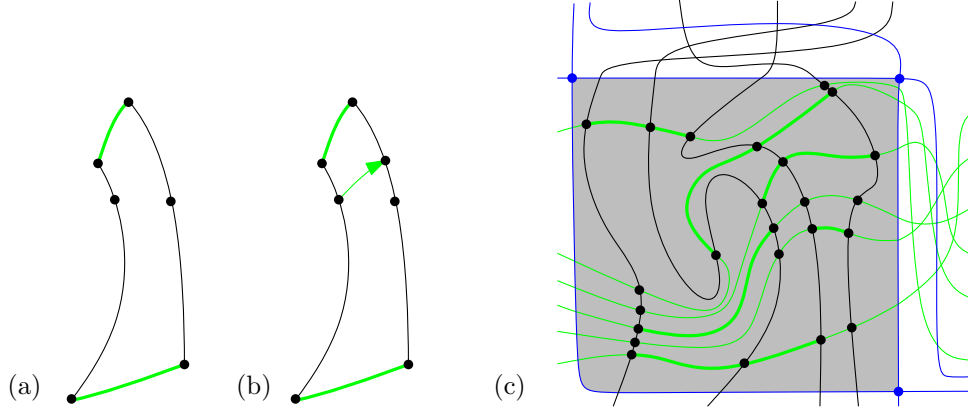


Figure 12: (a) A quadrilateral face with extra vertices; the shaded face from Fig. 11 (a). (b) Adding an edge. (c) Embedding the grid into a pseudoline arrangement.

Now we are almost done: each horizontal curve extends from v_1 to v_m and crosses each vertical curve exactly once. Now we just extend the horizontal curves to infinity, left of v_1 , and right of v_m , without crossings. ☺

This lemma can be interpreted in the context of pseudoline arrangements. In an arrangement of pseudolines, each pseudoline is an unbounded Jordan curve, and every pair of pseudolines has to cross *exactly* once. The grid construction can be embedded in a true pseudoline arrangement, see Fig. 12 (c): simply enclose all crossings in a bounded region formed by three new (blue) pseudolines and let the crossings between vertical lines and between horizontal lines occur outside this region.

We return to the proof of Theorem 14.

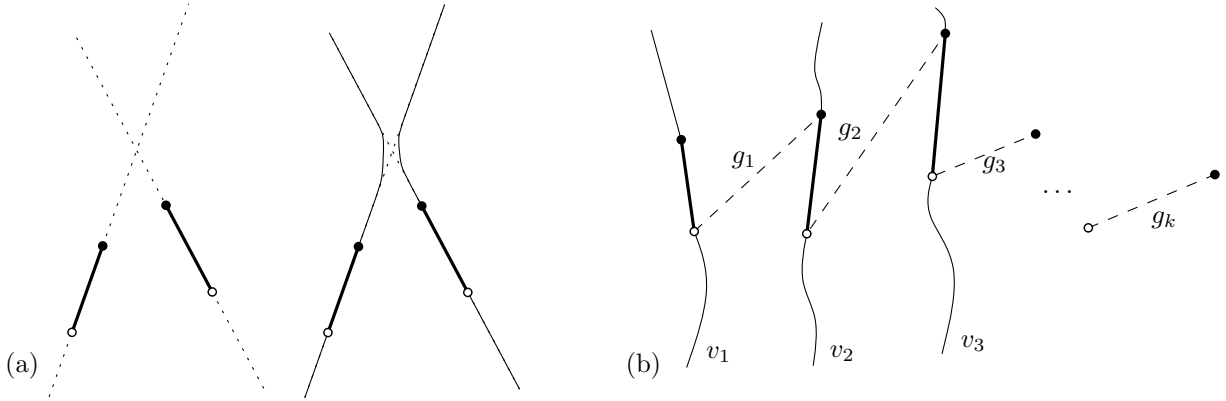


Figure 13: Applying the Fishnet Lemma.

Proof. Given a quasi-parallel matching M we construct a set of Jordan curves V as in Lemma 16 by considering the line arrangement formed by the segments $s_1 \triangleleft \dots \triangleleft s_n$ with the corresponding lines $g(s_1), \dots, g(s_n)$. We construct curve v_i by going along $g(s_i)$. At each intersection, the curves switch from one line to the other, and after a slight deformation in the vicinity of the intersections, they become non-crossing, see

Fig. 13 (a). These crossings lie outside the parts of the lines where the segments lie; therefore the switching have no influence on the left-to-right order of the segments s_i ;

Now assume there is another matching M' ; M and M' form at least one alternating cycle. Let $G = \{g_1, \dots, g_k\}$ be the segments of M' on such a cycle in the order in which they are traversed. V and G satisfy the condition of the Fishnet Lemma and thus can be extended to a full combinatorial grid. Assume without loss of generality that g_1 is above g_2 on the common incident edge of M . Then g_2 must also lie above g_3 , and so on. It follows that the extended horizontal curves h_1, \dots, h_k also must be in this order, and g_k would lie below g_1 . So they cannot be connected to the same segment in M . \nmid ☺

5 Circular Matchings

In this section we study circular matchings in more detail. Recall that such a matching is a BR-matching without a chromatic cut for which all points on the convex hull have the same color. We assume without loss of generality that this color is \bullet .

We prove that if M is of circular type, then its point set has at least two other matchings. Moreover, we show that for a circular matching, the relation \triangleleft induces a *circular order* (this will justify the term “matching of circular type”), and describe such matchings in terms of forbidden patterns.

Lemma 17. *A BR-matching M is of circular type if and only if it has no patterns (a) and (b) from Fig. 2, and has at least one pattern (c) (a “3-star”).*

Proof. We saw in Lemma 4 that a BR-matching has no chromatic cut if and only if it avoids the patterns (a) and (b). By Lemma 7, a BR-matching without chromatic cut is either of type L or of type C. By Lemma 13, a BR-matching is of type L if and only if it avoids (a), (b) and (c). Therefore, a BR-matching is of type C if and only if it avoids (a) and (b), but contains (c). ☺

Theorem 18. *Let M be a matching of circular type on the point set F . Then there exist (at least) two disjoint BR-matchings on F , compatible to M .*

Proof. According to Lemma 17 there are segments in the 3-star configuration as in Fig. 2. They partition the plane into three convex regions Q_1 , Q_2 and Q_3 and a triangle as in Fig. 14 (a). The triangle is bounded (without loss of generality) by three \circ -rays, and it is empty of segments: indeed, any segment of M inside the triangle would emit a \bullet -ray, which would cross a \circ -ray — a contradiction to the fact that M has no chromatic cut, see Lemma 4.

All segments in a region Q_i together with the two defining segments are of linear type (indeed, they have no chromatic cut but have both colors on the boundary of the convex hull). Thus, by Lemma 10, in each region there is an alternating path from the \bullet -point of the left bounding segment to the \circ -point of the right bounding segment (or vice versa). The union of the three paths forms an alternating polygon and thus we have found a different compatible BR-matching M' . If we choose the paths in the other direction (\circ -point of the left bounding segment to the \bullet -point of the right bounding segment), we get another BR-matching M'' . ☺

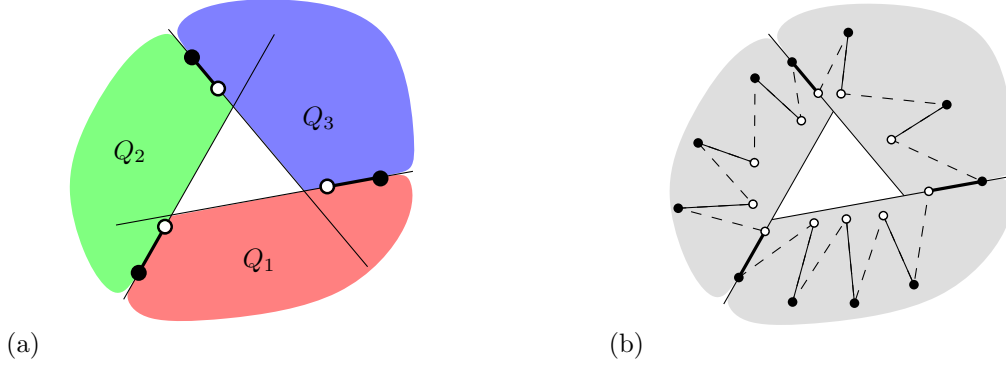


Figure 14: Illustration to the proof of Theorem 18.

Now we study in more detail the relation \triangleleft for circular matchings. In the proof of Theorem 18 we saw that a circular matching is a union of three linear matchings (see Fig. 14 (b)). In the next Proposition we prove that in fact it is a union of *two* linear matchings.

Lemma 19. *Let M be a circular matching, and let B be a segment of M . The matchings*

$$\begin{aligned} M_B^R &= \{X \in M : B \triangleleft X\}, \\ M_B^{R+} &= \{X \in M : B \triangleleft X\} \cup \{B\}, \\ M_B^L &= \{X \in M : X \triangleleft B\}, \\ M_B^{L+} &= \{X \in M : X \triangleleft B\} \cup \{B\}. \end{aligned}$$

are not empty, and they are of linear type.

Proof. Consider first the matching M_B^{R+} . Since it contains B , it is non-empty. Since it is a submatching of M , it has no chromatic cut. Both the \circ - and the \bullet -end of B belong to the boundary of its convex hull; therefore M_B^{R+} must be of linear type. Similarly, M_B^{L+} is of linear type.

If M_B^R is empty, then $M_B^{L+} = M$, which is impossible since M is of circular type, and M_B^{L+} of linear type. Now, since M_B^{R+} is of linear type, and M_B^R is a subset of this matching, M_B^R is of linear type as well (see the remark after Theorem 2). The proof for M_B^L is similar. \odot

Corollary 20. *The relation \triangleleft in a matching of circular type has no minimum or maximum element:*

$$\begin{aligned} \forall B: \exists A: A \triangleleft B \\ \forall B: \exists A: B \triangleleft A \end{aligned}$$

Proof. For such an element B , M_B^L or M_B^R would be empty. \odot

Lemma 21. *Let M be a circular matching. Let B be any segment of M . Let A be the minimum (with respect to \triangleleft) element of M_B^L , and let Z be the maximum element of M_B^R . Then the triple $\{A, B, Z\}$ is a circular matching (a 3-star).*

Proof. If M is of size 3, that is, $M = \{A, B, Z\}$, there is nothing to prove. So, we assume that there is at least one more segment in M . Assume without loss of generality that M_B^R contains at least one segment in addition to Z .

Let D be a segment of M such that $D \triangleleft A$ (such a segment exists by Proposition 19). Since A is the minimum element of M_B^L , we have $D \in M_B^R$, that is, $B \triangleleft D$.

If $D = Z$ then we have $Z \triangleleft A \triangleleft B \triangleleft Z$: that is, the relation \triangleleft in the triple $\{A, B, Z\}$ is not linear; therefore $\{A, B, Z\}$ is of circular type.

Suppose now that $D \neq Z$, and consider the matching $\{A, B, D, Z\}$. We have $D \triangleleft A \triangleleft B \triangleleft D$. So, the relation \triangleleft in the matching $\{A, B, D, Z\}$ is not linear; therefore, $\{A, B, D, Z\}$ is of circular type. Now, by Lemma 19, some segment in $\{A, B, D, Z\}$ must lie to the right of Z . Since $B \triangleleft Z$ and $D \triangleleft Z$, we have $Z \triangleleft A$. So, we have $Z \triangleleft A \triangleleft B \triangleleft Z$, and this means that $\{A, B, Z\}$ is of circular type. \odot

We shall show that if M is a circular matching, then there exists a natural *circular order* of its members. A circular (or cyclic) order is a ternary relation which models the “clockwise” relation among elements arranged on a cycle. A standard way of constructing a circular order from j linear orders $A_{11} \leq A_{12} \leq \dots \leq A_{1i_1}$, $A_{21} \leq A_{22} \leq \dots \leq A_{2i_2}$, \dots , $A_{j1} \leq A_{j2} \leq \dots \leq A_{ji_j}$ is their “gluing”: we say that $[X, Y, Z]$ (and, equivalently, $[Y, Z, X]$ and $[Z, X, Y]$) if we have $X \leq \dots \leq Y \leq \dots \leq Z$ in the sequence

$$A_{11} \leq A_{12} \leq \dots \leq A_{1i_1} \leq A_{21} \leq A_{22} \leq \dots \leq A_{2i_2} \leq \dots \leq A_{j1} \leq A_{j2} \leq \dots \leq A_{ji_j} \leq A_{11}$$

In this line \leq relates only to pairs of neighbors; in particular, it is not transitive in this line.

We fix $B \in M$ and apply this procedure on M_B^{L+} and M_B^R in which \triangleleft is linear by Lemma 19. Let A_1, A_2, \dots, A_k be the segments of M_B^L labeled so that $A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_k$, and let C_1, C_2, \dots, C_m be the segments of M_B^R labeled so that $C_1 \triangleleft C_2 \triangleleft \dots \triangleleft C_m$. By Lemma 21 we have $C_m \triangleleft A_1$. Thus, we consider the circular order $[*, *, *]$ induced by

$$B \triangleleft C_1 \triangleleft C_2 \triangleleft \dots \triangleleft C_m \triangleleft A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_k \triangleleft B. \quad (1)$$

That is, for $X, Y, Z \in M$ we have $[X, Y, Z]$ (and, equivalently $[Y, Z, X]$ and $[Z, X, Y]$) if and only if we have in (1) $X \triangleleft \dots \triangleleft Y \triangleleft \dots \triangleleft Z$, or $Y \triangleleft \dots \triangleleft Z \triangleleft \dots \triangleleft X$, or $Z \triangleleft \dots \triangleleft X \triangleleft \dots \triangleleft Y$. Notice that we always have either $[X, Y, Z]$ or $[X, Z, Y]$ (but never both).

The circular order $[*, *, *]$ will be referred to as the *canonical circular order* on M . The next results describe the geometric intuition beyond this definition: we shall see that $[X, Y, Z]$ means in fact that these segments appear in this order clockwise. Moreover, we shall see that the definition of $[*, *, *]$ does not depend on the choice of B .

Lemma 22. *Let M be a circular matching, and let $X, Y, Z \in M$. Then we have $[X, Y, Z]$ if and only if at least two among the following three conditions hold: $X \triangleleft Y$; $Y \triangleleft Z$; $Z \triangleleft X$.*

If all three conditions hold, then $\{X, Y, Z\}$ is a 3-star; and if exactly two among the statement hold, then $\{X, Y, Z\}$ is a linear matching. All possible situations for $[X, Y, Z]$ (with respect to \triangleleft) appear in Fig. 15.

Proof. The segment B from the definition of $[*, *, *]$ is the maximum element of M_B^{L+} . Therefore, it is convenient to denote $A_{k+1} = B$. Now we have four cases.

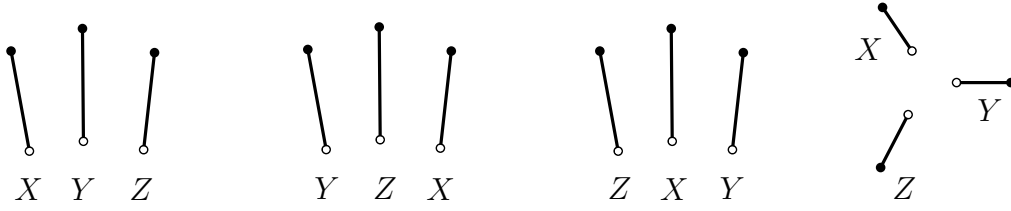


Figure 15: Possible configurations of three segments that satisfy $[X, Y, Z]$.

- Case 1: $X, Y, Z \in M_B^{L+}$.

In this case $\{X, Y, Z\}$ is of linear type (by Lemma 19). Therefore either one or two of the conditions hold. If exactly two conditions hold: assume without loss of generality that $X \triangleleft Y \triangleleft Z$. Since $A_1 \triangleleft \dots \triangleleft A_{k+1}$ is a linear order in M_B^{L+} , we have $X = A_\alpha, Y = A_\beta, Z = A_\gamma$ for some $1 \leq \alpha < \beta < \gamma \leq k+1$. Now we have $[X, Y, Z]$ by definition. If exactly one condition holds: assume that it is $X \triangleleft Y$; then we have $Y \triangleleft Z \triangleleft X$, which implies “not $[X, Y, Z]$ ”.

- Case 2: two members of $\{X, Y, Z\}$ belong to M_B^{L+} , and one to M_B^R . Assume without loss of generality that $X, Y \in M_B^{L+}, Z \in M_B^R$ and that $X \triangleleft Y$.

Then we have $X = A_\alpha, Y = A_\beta$ for some $\alpha < \beta$ and $Z = D_\gamma$ for some γ , and so $[X, Y, Z]$.

At the same time in this case at least two of the conditions hold: indeed, assume $X \triangleleft Z \triangleleft Y$. Then B is distinct from X, Y, Z (in particular, $B \neq Y$ because $B \triangleleft Z$). Now, in the matching $\{X, Y, Z, B\}$ there is a minimum element, X , but there is no maximum element. Therefore, $\{X, Y, Z, B\}$ is neither of linear nor of circular type — a contradiction.

- Case 3: one member of $\{X, Y, Z\}$ belongs to M_B^{L+} , and two to M_B^R , and Case 4: all the members of $\{X, Y, Z\}$ belong to M_B^R , are similar to cases 2 and 1. Therefore, we omit their proofs.

☺

Corollary 23. *The canonical circular order doesn't depend on the choice of B .*

Proof. Indeed, if another choice of B gave another circular order, there would be a triple that belongs to one of them and doesn't belong to another. However, in Lemma 22 we saw an equivalent definition that only depends on relations between triples of segments. ☺

Lemma 24. *Let M be a circular matching, and let $X \in M$. Then the immediate successor of X the canonical circular order is the minimum element of M_X^{R+} .*

Proof. This is immediate for B (as in definition of $[\ast, \ast, \ast]$), and, since we saw in Corollary 23 that the circular order $[\ast, \ast, \ast]$ does not depend on the choice of B , this is true for all segments. ☺

Lemmas 22 and 24 show that the canonical circular order describe the combinatorial structure of circular matchings in a natural way, similarly to that in which \triangleleft describes the structure of linear matchings. In Subsection 7.2 we'll provide a finer classification of relations \triangleleft realizable in circular matchings.

6 Summary of the Proof of the Characterization Theorem 2 about Unique BR-Matchings and Theorem 3 about Circular Matchings

We summarize the proofs of both Theorems.

We start with the equivalence of all five conditions in Theorem 2. Equivalence of conditions 2, 3, 4, 5 is proven in Lemma 13. Finally, $2 \Rightarrow 1$ (if a BR-matching M is of linear type, then it is unique) is proven in Theorem 14; and $1 \Leftrightarrow 2$ (if a BR-matching M is unique, then it is of linear type) follows from Corollary 6 (if M is unique, then it has no chromatic cut), Lemma 7 (if M has no chromatic cut, then it is either of linear or circular type), and Theorem 18 (if M is of circular type, then it is not unique).

Consider the conditions 1, 2, 3 and Properties p1 and p2 in Theorem 3. All three conditions imply that M has no chromatic cut and exclude that M is of linear type. Thus in all cases M must be a circular matching. Property p1 is explained in Section 5 and p2 is proved by Theorem 18.

7 Miscellaneous

A *parallel matching* is a BR-matching that consists of parallel segments. As we saw in Theorem 2, quasi-parallel matchings generalize parallel matchings in the sense that they are exactly the BR-matchings for which the relation \triangleleft is a linear order. Similarly, circular matchings generalize *radial matchings* – BR-matchings whose members lie on distinct rays with a common endpoint O and oriented away from O .

In this section we study how far quasi-parallel (resp., circular) matchings generalize parallel (resp., radial) matchings, in two aspects. In Subsection 7.1 we consider order types, and in Subsection 7.2 we deal with \triangleleft relations realizable in such matchings.

7.1 Order types in parallel vs. quasi-parallel matchings

Since, as mentioned above, quasi-parallel matchings generalize parallel matchings, it is natural to ask whether all order types (determined by orientations of triples of points) of bichromatic point sets with a unique BR-matching are realizable by corresponding endpoints of a parallel matching.

We construct an example that shows that the answer to this question is negative. The construction is based on the following observation.

Observation 25. *Let A, B, C be three parallel vertical segments such that $A \triangleleft B \triangleleft C$. Denote by a_1, b_1, c_1 the upper ends, and by a_2, b_2, c_2 the lower ends of the corresponding segments. If*

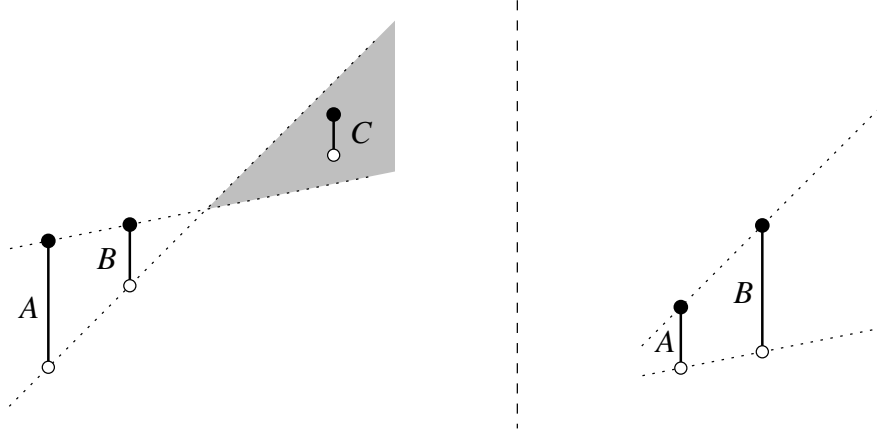


Figure 16: Illustration to Observation 25.

the triple $[a_1, b_1, c_2]$ is oriented counterclockwise, and the triple $[a_2, b_2, c_1]$ clockwise, then B is shorter than A .

Proof. The conditions mean that c_2 is situated above the line a_1b_1 , and c_1 below the line a_2b_2 . However, if B is not shorter than A , then the wedge that should contain C is situated to the left of A . Thus, $A \triangleleft C$ is impossible. See Fig. 16 for illustration. ☺

Now, the construction goes as follows. Consider three pairs of parallel (auxiliary) lines with slopes, say, 0° , 60° , and 120° , and three vertical segments A_0, B_0, C_0 , as shown in Fig. 17a. Change slightly the slopes of the lines so that each pair will intersect as indicated schematically in the right part, and so that the new segments A, B, C whose endpoints are intersection points of the modified lines are almost vertical. Add vertical segments in the wedges formed by the auxiliary lines, as shown in Fig. 17b. This can be done so that the new matching (consisting of six segments) is quasi-parallel; denote it by M .

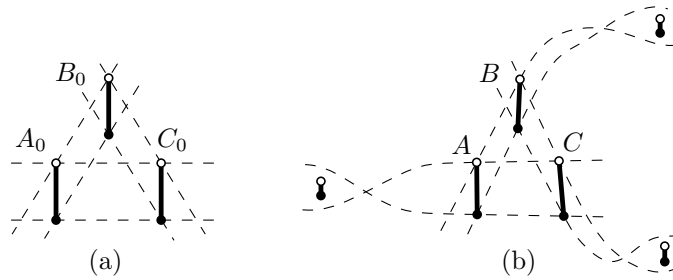


Figure 17: The construction of a “non-parallelizable” quasi-parallel matching.

Now, assume that there exists a parallel matching M' with endpoints of the same order type, and denote by A', B', C' the segments that correspond in M' to A, B, C . Then, according to Observation 25, A' is longer than B' , B' is longer than C' , and C' is longer than A' . This is a contradiction. ☺

In the same manner radial matchings, as defined above, do not capture the order type. To see this let M be a non-parallelizable linear BR-matching and assume there

exists a radial representation M' with the same order type and apex O . Project the point O to infinity with a projective mapping such that the matching becomes parallel. As projective mappings conserve the order type we parallelized M – a contradiction. \odot

7.2 Sidedness relation in circular matchings

If $M = \{A_0, A_1, \dots, A_{n-1}\}$ is a linear matching,¹ and we know that $A_0 \triangleleft A_1 \triangleleft \dots \triangleleft A_{n-1}$, then the relation \triangleleft is completely determined (since it is linear by Lemma 13). In contrast, for matchings of n segments of circular type, there are several relations \triangleleft that satisfy $A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_{n-1} \triangleleft A_0$. In Theorem 29 we enumerate such relations; its proof also provides us with a classification of circular matchings in the sense of relations \triangleleft realizable in such matchings.

Lemma 26. *Let M be a circular matching, and assume that its canonical circular order is induced by $A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_{n-1} \triangleleft A_0$. Fix $i \in \{0, 1, \dots, n-1\}$. Then there exist a unique $j \in \{0, 1, \dots, n-1\}$ such that A_j and $A_{j+1} \pmod n$ are separated by $g(A_i)$.*

Proof. Let A_j be the maximum member of $M_{A_i}^R$. Then we have $A_{j+1} \in M_{A_i}^L$, and, thus, A_j and A_{j+1} are separated by $g(A_i)$. For any other pair of neighboring segments, either both belong to $M_{A_i}^{L+}$ or to $M_{A_i}^{R+}$, and, therefore, are not separated by $g(A_i)$. \odot

The pair of segments (A_j, A_{j+1}) as in Lemma 26 will be called the *antipodal pair* of A_i . Notice that by Lemma 21 such A_i, A_j and A_{j+1} form a 3-star. We say that $A, B \in M$ are *twins* if they have the same antipodal pair. Clearly, being twins is an equivalence relation; the equivalence classes are the maximal sets of twins (we shall call them *T-sets*). In Fig. 20 we have five T-sets: $\{9, 0, 1\}$, $\{2\}$, $\{3, 4, 5\}$, $\{6, 7\}$, and $\{8\}$. It is easy to see that any T-set is a linear matching.

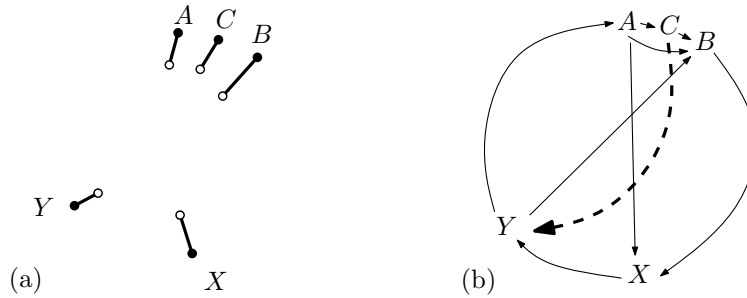


Figure 18: (a) Geometric intuition for Lemma 27. (b) The arrow indicates the sidedness relation \triangleleft . The dashed arrow is impossible.

Lemma 27. *Let M be a circular matching, and let $A, B \in M$ (assumed $A \triangleleft B$) be twins. Then any $C \in M$ such that $A \triangleleft C \triangleleft B$ is also a twin of A and B .*

¹ We denote the segments by A_0, \dots, A_{n-1} rather than by A_1, \dots, A_n because modular arithmetic will be used in this section.

Proof. Let (X, Y) be the antipodal pair of A and of B . Then we have $C \triangleleft B \triangleleft X \triangleleft Y \triangleleft A \triangleleft C$, $A \triangleleft X$, $Y \triangleleft B$ and $A \triangleleft B$ (see Fig. 18.). If $C \triangleleft Y$, then the matching $\{A, C, B, Y\}$ has a maximum (B) , but has no minimum — a contradiction. Thus, $Y \triangleleft C$, and, similarly, $C \triangleleft X$. Therefore, $g(C)$ separates X and Y . Now it follows from the uniqueness in Lemma 26 that (X, Y) is the antipodal pair for C . \odot

We say that a circular matching is *basic* if it has no twins, or equivalently, if all T-sets consist of one segment. We first classify \triangleleft relations of basic matchings.

Lemma 28. *Let M be a **basic** circular matching, with the circular order induced by $A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_{n-1} \triangleleft A_0$. Then, for each $A_i \in M$, we have $|M_{A_i}^L| = |M_{A_i}^R|$.*

Proof. Let $B \in M_{A_i}^R$, and let (X, Y) be the antipodal pair of B . We claim that $X, Y \in M_{A_i}^L$. Indeed, if $X, Y \in M_{A_i}^R$, then $\{A_i, X, Y\}$ is a linear matching, contradicting what was observed after the definition of the antipodal pair; and if $X \in M_{A_i}^R, Y \in M_{A_i}^L$ then (X, Y) is the antipodal pair of A_i , which is impossible since in such a case A_i and B are twins.

Assume for contradiction and without loss of generality that $|M_{A_i}^L| < |M_{A_i}^R|$. Then we have more segments in $M_{A_i}^R$ than their potential pairs of neighboring segments. Therefore, there exist distinct segments $B, C \in M_{A_i}^R$ that have the same antipodal pair, and thus, they are twins – a contradiction.

It follows that n is necessarily odd, and that $M_{A_i}^R = \{A_{i+1}, A_{i+2}, \dots, A_{i+\frac{n-1}{2}}\}$ and $M_{A_i}^L = \{A_{i-\frac{n-1}{2}}, \dots, A_{i-2}, A_{i-1}\} \pmod{n}$. In particular, this means that for basic circular matchings, the relation \triangleleft is determined uniquely by $A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_{n-1} \triangleleft A_0$. \odot

Fig. 19 shows basic matchings of sizes 3, 5, 7.

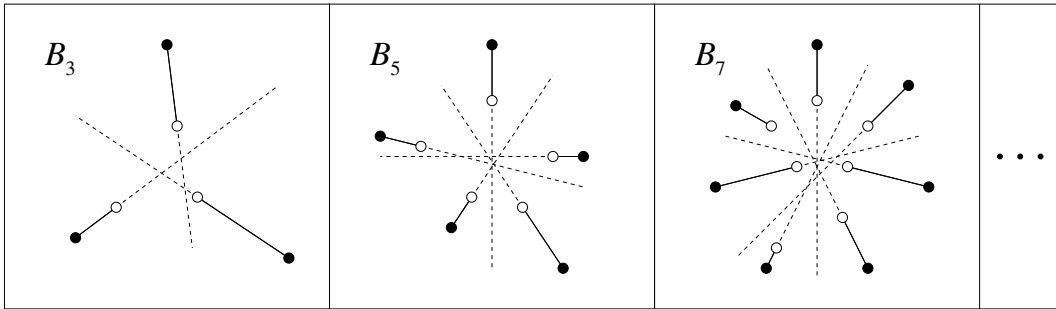


Figure 19: Basic matchings of sizes 3, 5, and 7.

Theorem 29. *The number of sidedness relations \triangleleft realizable by circular matchings $\{A_0, A_1, \dots, A_{n-1}\}$ that satisfy $A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_{n-1} \triangleleft A_0$ is $2^{n-1} - n$.*

Proof. Consider a circular matching M of size n . Assume that there are k T-sets. Choose one segment from each T-set as follows: in the T-set which contains A_0 , we choose A_0 ; in all other T-sets we choose the minimum element (with respect to \triangleleft). The chosen segments form a basic matching B_k . Recall that, by Lemma 28, the relation \triangleleft between the members of B_k is uniquely determined. Now M can be obtained from B_k by recovering the twins of the chosen segments. The relation \triangleleft for M is then only determined by sizes

of T-sets except that of A_0 ; and for the T-set of A_0 it also matters how many segments lie to the left of A_0 and how many to the right. Thus, we have $k + 1$ “regions” for adding twins, and this is the combinatorial problem of choosing a multiset of size $n - k$ from $k + 1$ elements. For fixed k , the corresponding generating function is

$$x^k(1 + x + x^2 + x^3 + \dots)^{k+1} = x^k \left(\frac{1}{1-x} \right)^{k+1},$$

the summation over all odd $k \geq 3$ gives

$$\frac{x^3}{(1-x)^4} + \frac{x^5}{(1-x)^6} + \frac{x^7}{(1-x)^8} + \dots = \frac{x}{1-2x} - \frac{x}{(1-x)^2}. \quad (2)$$

Since

$$\frac{1}{1-2x} = \sum_{n \geq 0} 2^n x^n \quad \text{and} \quad \frac{1}{(1-x)^2} = \sum_{n \geq 0} (n+1)x^n,$$

the coefficient of x^n in the generating function (2) is $2^{n-1} - n$. ☺

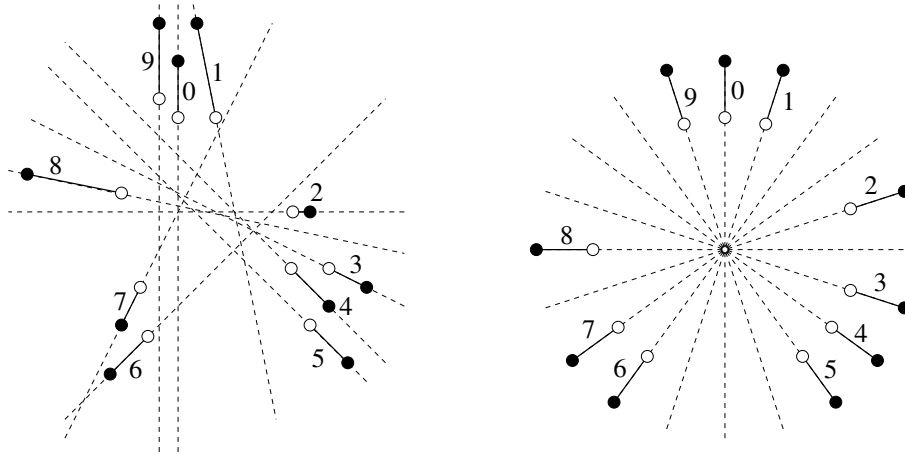


Figure 20: A circular matching and its “standardization”.

It is easy to see from the reasoning above that any \triangleleft relation which is realizable with circular matchings can be realized by a radial matching; furthermore, it is possible to take the lines such that the angle between each pair of adjacent lines will be π/n , and the endpoints the segments lie on two fixed circles with center O , see Fig. 20). Now the formula $2^{n-1} - n$ becomes especially clear: consider n lines passing through a common point O . For each line, there are two choices on which ray we put a segment (except the fixed A_0). Thus we have 2^{n-1} matchings: n of them are of linear type, and the others are of circular type.

7.3 Description in terms of point sets

We described point sets with unique matchings in terms of a given matching M rather than in terms of the set F itself. It would be nice to characterize the points sets F

directly, for example by forbidden patterns of *points*. However, such a characterization is impossible.

Suppose that there are is a collection of patterns of points (of two colors) such that F has unique matching if and only if F avoids these patterns. Equivalently, F has several matchings if and only if F contains any of these patterns. However, in such a case we can duplicate all the members of F : for each $p_i \in F$ we add a point p'_i so that p_i and p'_i are of opposite colors, all the segments $p_i p'_i$ are parallel (including orientation), and the new set is in general position (see Fig. 21). Then the matching that consists of all the segments $p_i p'_i$ is a (quasi-)parallel linear matching, and thus is a unique matching of the new set, while it contains the assumed pattern(s).

We can actually move the additional points as far away as we like. Thus, even a more “local” characterization, that a certain convex region should contain some pattern and no other points, is impossible.

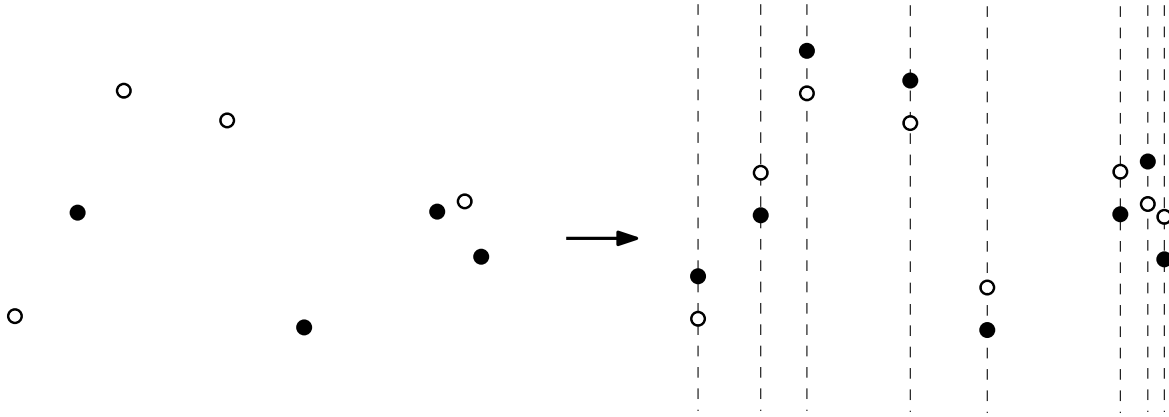


Figure 21: Illustration of the duplication described in Subsection 7.3.

On the other hand, suppose that there are is a collection of point patterns such that F has several matchings if and only if F avoids these patterns. Equivalently, F has unique matching if and only if F contains any of these patterns. However, in such a case we can take this matching and add one more segment to obtain a BR-matching with a chromatic cut. So, the new point set will have more than one matching while it contains the assumed pattern(s). As above, the additional segment can be placed arbitrarily far away.

8 Algorithms

In this section we describe several algorithms. The first checks whether a given point set F has a unique BR-matching. This algorithm is based on yet another characterization of unique BR-matchings. The second checks if a given BR-matching is circular. Applying these algorithms together, we can check if a given matching has a chromatic cut.

Definition 30. A BR-matching M has the *drum property* with respect to the segments $A, B \in M$ ($A \neq B$) if A and B are the only segments from M on $\partial\text{CH}(F)$.

Theorem 31. Let $M = \{A_1, A_2, \dots, A_n\}$ be a BR-matching such that $A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_n$. Then the following conditions are equivalent:

1. M is the unique BR-matching.
2. For every $i < j$, every subset $S \subseteq \{A_i, A_{i+1}, \dots, A_j\}$ with $A_i, A_j \in S$ has the drum property for A_i and A_j .
3. For every $j > 1$, the set $\{A_1, A_2, \dots, A_j\}$ has the drum property for A_1 and A_j ; and for every $i < n$, the set $\{A_i, A_{i+1}, \dots, A_n\}$ has the drum property for A_i and A_n .

Note that the relation \triangleleft is not necessarily transitive. So the assumption of the theorem does not imply $A_i \triangleleft A_j$ for $i < j$.

Proof. “1 \Rightarrow 2” follows directly from Theorem 2, Condition 2, together with the remark after the theorem that the condition is implied for all subsets.

“2 \Rightarrow 3” is clear.

“3 \Rightarrow 1”: Since $\{A_1, A_2, \dots, A_j\}$ has the drum property for A_1 and A_j , all segments A_1, \dots, A_{j-1} lie on the same side of $g(A_j)$. Since $A_{j-1} \triangleleft A_j$ by assumption, we know that the segment A_{j-1} lies left of A_j , and hence we conclude that all segments A_i lie left of $g(A_j)$, for $i < j$. Similarly, from the drum property for $\{A_i, A_{i+1}, \dots, A_n\}$ we conclude that the segments A_j lie right of $g(A_i)$, for $j > i$. These two conditions together mean that $A_i \triangleleft A_j$ for $i < j$. Therefore, Condition 3 of Theorem 2 holds, and M is unique. \odot

From Property 3 of Theorem 31 we can derive a linear-time algorithm for testing whether M is unique, once the ordering $A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_n$ has been computed: We incrementally compute $P_j := \text{CH}(\{A_1, A_2, \dots, A_j\})$ for $j = 2, \dots, n$ and check the drum property as we go.

There is a straightforward incremental algorithm for computing the convex hull (see, for example, [8]), which is the basis for more elaborate randomized incremental algorithms that work also in higher dimensions, see [4, Chapter 11]. It extends a convex hull C by a new point p as follows:

- C1. Check whether $p \in C$. If this is the case, stop.
- C2. If not, find a boundary point $q \in \partial C$ that is visible from p .
- C3. Walk from q in both directions to find the tangents pq_1 and pq_2 from p to C .
- C4. Update the convex hull: remove the part between q_1 and q_2 that has been walked over, and replace it with q_1pq_2 .

Steps C3 and C4 take only linear time overall, because everything that is walked over is deleted. The “expensive” steps that are responsible for the superlinear running time of convex hull algorithms are C1 and C2. However, in our case, we will see that these steps are trivial. (We extend the convex hull by inserting not a single point but two points of A_{j+1} at a time.)

Since the drum property holds for $\{A_1, A_2, \dots, A_{j+1}\}$ we know that the new points of A_{j+1} don’t lie in P_j , and since A_j lies on the boundary of P_j but not of P_{j+1} , we know that A_j is visible from the points of A_{j+1} . We can start the search in step C3 from there.

This visibility assumption can be checked (as part of checking the drum property) in constant time. The overall running time is linear.

In a second symmetric step, we start from the end and compute $\text{CH}(\{A_i, A_{i+1}, \dots, A_n\})$ for $i = n - 1, \dots, 1$.

Theorem 32. *It can be checked in $O(n \log n)$ time whether a bichromatic $(n + n)$ -set has a unique non-crossing BR-matching.*

Proof. First we have to compute some BR-matching $M = \{A_1, A_2, \dots, A_n\}$. It is well-known that this can be done by recursive ham-sandwich cuts in $O(n \log n)$ time. A ham-sandwich cut is a line ℓ that partitions a bicolored $(n + n)$ -set such that each open half-plane contains at most $\lfloor \frac{n}{2} \rfloor$ points of each color. If n is odd, ℓ must go through a red and a blue point. We can match these points to each other, and recursively find a BR-matching in the $(\frac{n-1}{2} + \frac{n-1}{2})$ -sets in each half-plane. If n is even, ℓ may go through one or two points, but by shifting ℓ slightly we can push these points to the correct side such that each half-plane contains an $(\frac{n}{2} + \frac{n}{2})$ -set. We recurse as above. A ham-sandwich cut can be found in linear time [10]. Hence this procedure leads to a running time of $T(n) = O(n) + 2 \cdot T(n/2)$, which gives $T(n) = O(n \log n)$.

Next, we compute an ordering

$$A_1 \triangleleft A_2 \triangleleft \dots \triangleleft A_n. \quad (3)$$

We do this by a standard sorting algorithm in $O(n \log n)$ time, assuming that the relation \triangleleft is a linear order. If, at any time during the sort, we find two segments that are not comparable by \triangleleft , we quit. Finally, we check condition (3) in $O(n)$ time. (This final check is not necessary, if, for example, mergesort is used as the sorting algorithm.)

As the last step, we check Property 3 of Theorem 31 in linear time, as outlined above. ☺

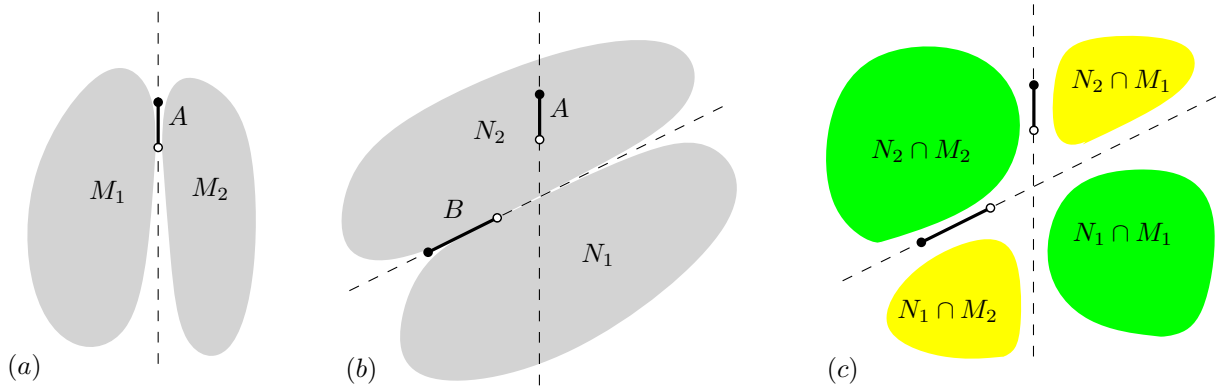


Figure 22: Separation of M into 6 overlapping BR-Matchings

It is also possible to determine in $O(n \log n)$ if a BR-matching M is circular, by an easy divide-and-conquer algorithm. Let A and B be two arbitrary segments in M . Let

$M_1 = M_A^{L+}$ and $M_2 = M_A^{R+}$ (that is, the segments that lie to the left or to the right of A , including A itself) and likewise $N_1 = M_B^{L+}$ and $N_2 = M_B^{R+}$ (see Fig. 22; recall that the segments are implicitly directed from white to black). M_i and N_i are linear matchings by Lemma 19. Finally, define $Q_1 := (M_2 \cap N_2) \cup (M_1 \cap N_1)$ and $Q_2 := (M_1 \cap N_2) \cup (M_2 \cap N_1)$.

Observation 33. *A BR-matching M has a chromatic cut if and only if at least one of the six matchings defined above has a chromatic cut.*

Proof. Consider two segments in M . Then they must be both in one of the matchings M' as defined above. If they have a chromatic cut, then M' has a chromatic cut. The other direction is obvious. ☺

Theorem 34. *It can be checked in $O(n \log n)$ time whether a BR-matching M is of circular type.*

Proof. The algorithm starts to compute the convex hull of M . If all points on $\partial CH(M)$ are of the same color we know it is not a linear matching and it remains to check if M has no forbidden pattern as in Fig. 2 (a)-(b).

We pick any segment A_0 and split M along the supporting line of A_0 . We compute the linear order of both parts. This gives a potential circular order. We remember this order for the remaining part.

The rest of the algorithm works recursively. We start by defining M_1 and M_2 as above to any segment A . Let B be the median of the larger of the M_i with respect to \triangleleft . The BR-matchings N_i and Q_i are also defined as above. For the BR-matchings M_i and N_i , it can be checked in linear time if they are linear, because we have already precomputed the order. As B is the median of the larger of the M_i , $n/4 \leq |Q_i| \leq 3n/4$, $\forall i$. We check recursively if Q_1 and Q_2 has no chromatic cut. For the running time $T(n)$, we have $T(n) \leq O(n) + \max_{1/4 \leq \alpha \leq 3/4} [T(\alpha n) + T((1 - \alpha)n)]$. Thus $T(n) = O(n \log n)$.

If any of these steps in the algorithm fails, a forbidden configuration is present. In this case we just stop and return that M has a chromatic cut. Otherwise we return the correct circular order. ☺

The last algorithm we want to present is about computing a balanced line as in Lemma 5. As a preprocessing step we need to find a point on a segment in general position with respect to the remaining points F .

Lemma 35. *Let F be a point set in the plane and $A = (a, b)$ be a vertical segment such that $F \cup \{a, b\}$ lies in general position (i.e., no three points on a line). Then the lowest intersection p of A with a segment formed by two points in F can be computed in deterministic $O(n \log n)$ time.*

Proof. Consider the point sets G and H left and right of $g(A)$. Let m_G be the median of the larger set G , with respect to the order defined by a ray rotating around b . The line k through m_G and b defines the four sets G_{up} , G_{low} , H_{up} and H_{low} , as in Fig. 23b. Now any two points defining the lowest intersection with C are either in G_{up} and H_{up} , or in two opposite sets (i.e., G_{up} and H_{low} or G_{low} and H_{up}). The lowest intersecting segment of G_{up} and H_{up} is the convex hull edge of G_{up} and H_{up} intersecting the supporting line $g(A)$. It can be found in linear deterministic time with a subroutine of the convex hull algorithm by Kirkpatrick and Seidel [9] or by

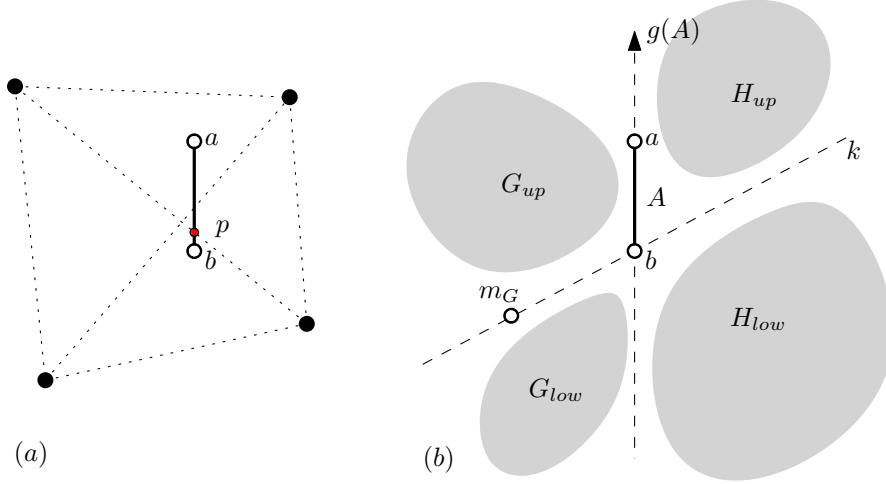


Figure 23: The line supporting line $g(A)$ splits the point set F into G and H .

an algorithm by Aichholzer, Miltzow and Pilz [2]. The second algorithm only uses order type information. The two opposite sets are treated recursively. Note that $n/4 \leq \#(G_{\text{low}} \cup H_{\text{up}}) \leq 3n/4$ and likewise $n/4 \leq \#(G_{\text{up}} \cup H_{\text{low}}) \leq 3n/4$. Therefore, for the running time we get $T(n) = O(n) + \max_{1/4 \leq \alpha \leq 3/4} [T(\alpha n) + T((1 - \alpha)n)]$ and $T(n) = O(n \log n)$. ☺

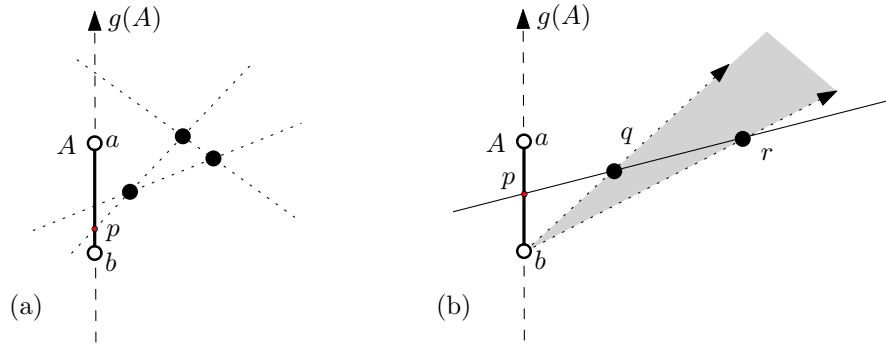


Figure 24: (a) the line arrangement formed by the points in F and the lowest crossing with A ; (b) the cone with apex b spanned by the minimal pair of points is empty of points of F

For the next Lemma we refer to Fig. 24.

Lemma 36. *Let A be a segment and F a point set right of $g(A)$ in general position. Then the lowest intersection p of A with a line through two points in F can be computed in deterministic $O(n \log n)$ time.*

Proof. First consider the points $q, r \in F$ which form the lowest crossing with A . We show they are neighbors in the radial order around b . Consider the area swept by a ray from q to r . If it contained any point s then either the line through q and s or r and s would have a lower intersection with A .

Thus we merely compute the radial order around b and for any neighboring pair the intersection point with A . The running time $T(n) = O(n \log n)$ is dominated by the sorting procedure. ☺

Theorem 37. *Let F be a point set in the plane and $A = (a, b)$ be a vertical segment such that $F \cup \{a, b\}$ lies in general position (i.e., no three points on a line). Then the lowest intersection of A with a line through two points in F can be computed in deterministic $O(n \log n)$ time.*

Proof. Compute the lowest intersection point with a line separately for the points left and right of A according to Lemma 36 and all possible intersections with A by pairs of points on opposite sites of A according to Lemma 35. ☺

Corollary 38. *Given a point set F and a segment A without three points on a line, a point on A in general position with respect to F can be computed in $O(n \log n)$ time.*

Proof. Any point between the lowest intersection and the lower endpoint of A is in general position with respect to F . ☺

Lemma 39. *Let M be a BR-matching of a point set F in general position and A, B be two segments as in Fig. 2. Then we can compute a balanced line through the interior of A or B in $O(n \log n)$ time.*

Proof. Let $p \in A$ and $q \in B$ be points as in Corollary 38. We know by the proof of Lemma 5 that a balanced line through p or q exists. The algorithm in [2] can be adapted to find the desired balanced line in through p or q in $O(n)$. ☺

Remark. Once we have $O(n \log n)$ algorithms to test whether a BR-matching is linear or circular we automatically receive an algorithm to test if a BR-matching has a chromatic cut in $O(n \log n)$. Note that both algorithms above can be executed till they find a forbidden configuration. Thus we are able to compute a forbidden configuration also in $O(n \log n)$. In the case of linear matchings we compute the linear order and for circular matchings the circular order. It is also easy to construct a reference line in linear time, as in the Definition 12 of quasi-parallel segments. Given a forbidden configuration it is also possible to compute in constant time a chromatic cut (i.e., the actual line). Finally, given a forbidden configuration, we can compute a balanced line intersecting one of the segments. In summary, *all* defined terms presented can be computed efficiently.

9 Open questions, Lower Bounds, etc.

Our algorithm for testing whether a point set F has a unique non-crossing BR-matching starts by finding such a BR-matching M , in $O(n \log n)$ time, by repeated ham-sandwich cuts. This algorithm does not care whether M is unique, and it is in fact the fastest known algorithm for finding *any* non-crossing BR-matching in an arbitrary point set. Is there a faster algorithm for checking whether M is unique (necessarily without constructing M)?

The paper just read could also be seen as the study of point sets with certain forbidden patterns. These particular point sets have a lot of nice geometric structure. We wonder whether also other forbidden patterns lead to interesting geometric properties.

Consider n blue, n red and n green points in \mathbb{R}^3 . By applying repeatedly ham-sandwich cuts we know that there exists a noncrossing colorful 3-uniform geometric matching. (Each edge is represented by the convex hull of its vertices.) Thus we ask for a geometric characterization of point sets with just one such matching.

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